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THE CONTINUOUSLY SUPPORTED RAIL SUBJECTED TO AN AXIAL FORCE AND A MOVING LOAD

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SUMMARY

The recent practice of welding railroad rails to each other suggests that considerable axial compression forces may be induced in the rails because of a raise in temperature. This in turn may reduce the critical velocity for the track to the range of operational velocities of modern high speed trains. The purpose of the paper is to demonstrate that this is indeed a possibility.

INTRODUCTION

The response of a continuously supported beam subjected to moving loads, was conducted first, in connection with the determination of stresses in railroad tracks, by S. Timoshenko [1]. The obtained results indicated that there exist a critical velocity, v_{cr}, at which the deflections become very large. For assumed values of a rail and the foundation parameter, Timoshenko found that v_{cr} is about 1200 miles per hour, about ten times larger than the highest speed of a locomotive at the time, and concluded that the static equations are sufficient for the analysis of stresses in railroad rails.

New technological problems, connected with the construction of high speed rocket test tracks [2], and the use of floating ice plates

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on water as a pavement for moving vehicles [3], accelerated the research activities in this area. A large number of these results, for beams, as well as plates, are discussed in a recent survey by A.D. Kerr [4].

The recent interest in high speed trains generated new interest in the response of continuously supported beams subjected to moving loads. In order to increase passenger comfort and at the same time decrease the wear of the rail and base, the individual rails are presently joined by welding the rail ends to each other [5]. Because of the lack of expansion joints, falling or rising temperatures may cause considerable axial compression or tension forces in the rail. It is reasonable to expect that an axial compression force may reduce v_{cr}; possibly to the range of operational velocities of modern high speed trains. The purpose of the present paper is to demonstrate that this is indeed a possibility.

It appears that there is no analysis which takes into consideration the effect of axial forces upon v_{cr} of continuously supported beams (see [4]). To study this effect, in the following we analyze an infinite beam which rests on a Winkler foundation, is subjected to a moving concentrated load P and a constant axial force N, as shown in Fig. 1. The investigation is based on the differential equation

$$EI \frac{\partial^4 w}{\partial x^4} + N \frac{\partial^2 w}{\partial x^2} + m \frac{\partial^2 w}{\partial t^2} + kw = P\delta(x, t) \tag{1}$$

where w is the lateral deflection, EI is the flexural rigidity of the beam, m is the mass of the beam per unit length of axis, and k is the foundation parameter.

THE PROPAGATION OF FREE WAVES

Before discussing the solution of equ. (1), we study first the propagation of free waves in the infinite beam; results needed later.

For this purpose we substitute the wave type expression

$$w(x,t) = w_o \sin\left[\frac{2\pi}{\lambda}(x-ct)\right] \tag{2}$$

into equ. (1) with P = 0, and obtain

$$\left[\left(\frac{2\pi}{\lambda}\right)^4 - \frac{N}{EI}\left(\frac{2\pi}{\lambda}\right)^2 - \frac{m}{EI}\left(\frac{2\pi}{\lambda}\right)^2 c^2 + \frac{k}{EI}\right] w_o \sin\left[\frac{2\pi}{\lambda}(x-ct)\right] = 0$$

Above equation is satisfied for any t and x when the term in the first bracket is zero, hence when

$$c = \sqrt{\frac{EI}{m}\left(\frac{2\pi}{\lambda}\right)^2 - \frac{N}{m} + \frac{k}{m}\Big/\left(\frac{2\pi}{\lambda}\right)^2} \tag{3}$$

Thus, expression (2) is a solution of the homogeneous equ. (1) when c satisfies equ. (3). It represents an infinite wave train with amplitude w_o, wave length λ, and propagation velocity c.

When N = 0, equ. (3) reduces to the one obtained by J. Dörr [6]. The corresponding graph is shown schematically in Fig. 2. The value $c_{min}\big|_{N=0}$ is obtained from the condition $[\partial c/\partial(2\pi/\lambda)]_{N=0} = 0$. It is easily found that it is located at

$$\left(\frac{2\pi}{\lambda}\right) = \sqrt[4]{\frac{k}{EI}} \tag{4}$$

and that

$$c_{min}\big|_{N=0} = \sqrt[4]{\frac{4kEI}{m^2}} \tag{5}$$

When c = 0, equ. (3) reduces to

$$N = EI\left(\frac{2\pi}{\lambda}\right)^2 + \frac{k}{\left(\frac{2\pi}{\lambda}\right)^2}$$

The graphical presentation of above equation is also shown in Fig. 2. The minimum value of N is obtained from the condition $[\partial N/\partial(2\pi/\lambda)]_{c=0} = 0$. It is found to take place for

$$\left(\frac{2\pi}{\lambda}\right) = \sqrt[4]{\frac{k}{EI}} \tag{6}$$

and that

$$N_{min}\Big|_{c=o} = 2\sqrt{kEI} = N_{cr} \tag{7}$$

the critical buckling load of the infinite beam [7].

Comparing (4) and (6), it follows that both minima are located at the same value of $(2\pi/\lambda)$. It may be easily shown that for any fixed $N \leqq N_{cr}$ the corresponding c_{min} is located at the $(2\pi/\lambda)$-value given in (4) or (6) and that the corresponding c_{min} is

$$c_{min} = \sqrt{\sqrt{\frac{4kEI}{m^2}} - \frac{N}{m}} \tag{8}$$

The above results suggest the rewriting of equ. (3) in the following non-dimensional form:

$$\frac{c}{c_{min}\Big|_{N=0}} = \sqrt{\frac{1}{2}\left(\gamma^2 + \frac{1}{\gamma^2}\right) - \frac{N}{N_{cr}}} \tag{9}$$

where

$$\gamma = \frac{2\pi}{\lambda}\sqrt[4]{\frac{k}{EI}} \tag{10}$$

The graphical presentation of equ. (9) is shown in Fig. 3.

It should be noted that, for each $N \leqq N_{cr}$, the propagation velocity c depends also upon the wave length λ and that the wave trains of the type (2) do exist only for $c \geqq c_{min}$. When the wave length $\lambda \to 0$ or $\lambda \to \infty$, the corresponding $c \to \infty$. From Fig. 3 it may also be seen that for a fixed $N \leqq N_{cr}$, to each $c > c_{min}$ there always correspond two waves with different wave length λ. That is, for a fixed $N < N_{cr}$, two waves, each with a different wave length λ, may propagate with the same velocity.

To the propagation velocity c_{min} there corresponds only one wave (2) with the wave length given in (4). An important feature of the results presented above is, that c_{min} decreases with increasing N, (i.e., with increasing compression force) and that c_{min} becomes zero at N_{cr}. With decreasing N, (i.e., with increasing tension force), c_{min} increases.

From equ. (8) it follows that for any $N > N_{cr}$ the propagation velocity c_{min} is a complex number. This is also the case for an $N < N_{cr}$ and any c value beyond the c = 0 curve shown in Fig. 2 and Fig. 3.

BEAM SUBJECTED TO A MOVING LOAD P

We consider now the response of a beam when it is subjected to a load P that moves at a constant velocity v_o, as shown in Fig. 1. The differential equation which describes the beam response is given in (1). Because of the infinite extent of the beam and base, and its constant properties, as well as because of the assumption that v_o = const., it appears reasonable to assume that after a time period the transient motions will become negligibly small and that the beam displacements will approach a steady state [4]. Thus, from the point of view of an observer which moves with the load, the deflections of the beam will appear static. This observation, suggests the possibility, often utilized before, of transforming the partial differential equation (1) into an ordinary differential equation in the moving reference frame (ξ, η, ζ) as shown in Fig. 1. With the new coordinates

$$\xi = x - v_o t \quad ; \quad \eta = y \quad ; \quad \zeta = z \tag{11}$$

differential equation (1) becomes

$$EI \frac{d^4 w}{d\xi^4} + (N + mv_o^2) \frac{d^2 w}{d\xi^2} + kw = P\delta(\xi) \tag{12}$$

Because of the steady state assumption, the load P moves at a constant altitude and hence does not experience an acceleration in the z-direction. Thus, P in equ. (12) represents only the static intensity of the load. Note that equ. (12) is identical to the equation of a beam which rests on a Winkler base, is compressed by an axial force $(N+mv_o^2)$, and is subjected to a lateral load P at $\xi = 0$.

The simplest method of solving equ. (12) is to set the right hand side equal to zero and to incorporate the concentrated load P through the matching conditions at $\xi = 0$. Setting

$$4\alpha^2 = \frac{N + mv_o^2}{EI} \quad ; \quad 4\beta^4 = \frac{k}{EI} \tag{13}$$

equ. (12) becomes

$$\frac{d^4 w}{d\xi^4} + 4\alpha^2 \frac{d^2 w}{d\xi^2} + 4\beta^4 w = 0 \tag{14}$$

Assuming $w = Ae^{s\xi}$ and substituting it into equ. (14), we obtain

$$s^4 + 4\alpha^2 s^2 + 4\beta^4 = 0 \tag{15}$$

The four roots of above equation are

$$s_{1,2,3,4} = \pm \sqrt{-2\left(\alpha^2 \mp \sqrt{\alpha^4 - \beta^4}\right)} \tag{16}$$

Noting that $\alpha^4-\beta^4 = (\alpha^2-\beta^2)(\alpha^2+\beta^2)$, we have to distinguish three cases:

$$\alpha^2 \lesseqqgtr \beta^2 \qquad \text{which corresponds to} \qquad v_o \lesseqqgtr \sqrt{\sqrt{\frac{4kEI}{m^2}} - \frac{N}{m}} \tag{17}$$

Note that the velocities v_o are real, only when

$$N < 2\sqrt{kEI} = N_{cr} \tag{18}$$

It is well known that, when $N > N_{cr}$, the stable beam shape is not straight and hence differential equation (1) is not applicable. For this case, the necessary analysis is very involved and is beyond the scope of the present paper.

In the following we analyze the case when $N < N_{cr}$, in order to determine the rail response when N approaches the buckling load N_{cr}; a simpler problem of more immediate practical significance, for which equ. (1) is applicable.

We consider the case when

$$v_o < \sqrt{\sqrt{\frac{4kEI}{m^2}} - \frac{N}{m}}$$

This corresponds to $\alpha^2 < \beta^2$. The roots of equ. (10) are complex numbers, namely

$$s_{1,2,3,4} = \pm \lambda + i\omega \tag{19}$$

where

$$\lambda = \sqrt{\beta^2 - \alpha^2} \quad ; \quad \omega = \sqrt{\beta^2 + \alpha^2} \tag{20}$$

The corresponding solution for the region ahead of the load P is

$$\begin{aligned} w_a(\xi) &= e^{+\lambda\xi} \left[A_1 \cos(\omega\xi) + A_2 \sin(\omega\xi)\right] + \\ &+ e^{-\lambda\xi} \left[A_3 \cos(\omega\xi) + A_4 \sin(\omega\xi)\right] \qquad \xi \geqq 0 \end{aligned} \tag{21}$$

and for the region behind the load

$$\begin{aligned} w_b(\xi) &= e^{+\lambda\xi} \left[A_5 \cos(\omega\xi) + A_6 \sin(\omega\xi)\right] + \\ &+ e^{-\lambda\xi} \left[A_7 \cos(\omega\xi) + A_8 \sin(\omega\xi)\right] \qquad \xi \leqq 0 \end{aligned} \tag{22}$$

It is reasonable to expect that as $\xi \to \infty$, $w_a = 0$, and as $\xi \to -\infty$, $w_b = 0$. Thus

$$A_1 = A_2 = 0 \quad ; \quad A_7 = A_8 = 0 \tag{23}$$

The remaining four constants are determined from the four conditions at $\xi = 0$

$$\left.\begin{aligned} & w_a(o) = w_b(o) && \left.\frac{dw_a}{d\xi}\right|_o = \left.\frac{dw_b}{d\xi}\right|_o \\ & \left.\frac{d^2 w_a}{d\xi^2}\right|_o = \left.\frac{d^2 w_b}{d\xi^2}\right|_o && \left(\frac{d^3 w_a}{d\xi^3} - \frac{d^3 w_b}{d\xi^3}\right)_o = \frac{P}{EI} \end{aligned}\right\} \tag{24}$$

It should be noted that in the last boundary condition the term involving

$(dw_a/d\xi - dw_b/d\xi)_o$ is missing, since according to the boundary condition above it, this term vanishes at $\xi = 0$. The determined constants are

$$
\begin{aligned}
A_4 &= - A_8 = \frac{P}{4EI\omega(\lambda^2 + \omega^2)} \\
A_3 &= + A_5 = \frac{P}{4EI\lambda(\lambda^2 + \omega^2)}
\end{aligned}
\tag{25}
$$

Substituting the constants into equations (21) and (22) we obtain, noting that $\xi = x - v_o t$,

$$
w_a(x,t) = \frac{Pe^{-\lambda(x-v_o t)}}{4EI\lambda\omega(\lambda^2+\omega^2)} \left\{\omega \cos[\omega(x-v_o t)] + \lambda \sin[\omega(x-v_o t)]\right\} \tag{26}
$$

$$\text{for } x \geqq v_o t$$

$$
w_b(x,t) = \frac{Pe^{+\lambda(x-v_o t)}}{4EI\lambda\omega(\lambda^2+\omega^2)} \left\{\omega \cos[\omega(x-v_o t)] - \lambda \sin[\omega(x-v_o t)]\right\} \tag{27}
$$

$$\text{for } x \leqq v_o t$$

Hence, the wave caused by P, and which moves with P at a constant velocity $v_o < \sqrt{\sqrt{4kEI/m^2} - N/m}$, is symmetrical with respect to P for any $N < N_{cr}$.

It may be easily shown that when v_o approaches the value $\sqrt{\sqrt{4kEI/m^2} - N/m}$, the deflections become infinite. Thus, the critical velocity of the beam subjected to the axial compression force N is

$$
v_{cr} = \sqrt{\sqrt{\frac{4kEI}{m^2}} - \frac{N}{m}} \tag{28}
$$

When N = 0, equ. (28) reduces to

$$
v_{cr}\Big|_{N=0} = \sqrt[4]{\frac{4kEI}{m^2}} \tag{29}
$$

Noting (29) and (7), equation (28) may be rewritten in the following form:

$$
\boxed{\frac{v_{cr}^2}{v_{cr}^2\big|_{N=0}} + \frac{N}{N_{cr}} = 1} \tag{30}
$$

or

$$v_{cr} = \sqrt{1 - N/N_{cr}}\ v_{cr}\Big|_{N=0} \tag{31}$$

Hence when $N \rightarrow N_{cr}$ the critical speed $v_{cr} \rightarrow 0$. When N is a tensile force, $v_{cr} > v_{cr}\big|_{N=0}$. The graphical presentation of equ. (31) is shown in Fig. 4.

Comparing equations (8) and (28) it follows that for any $N < N_{cr}$ $v_{cr} = c_{min}$. Thus, for a fixed $N < N_{cr}$, the corresponding v_{cr} is the lowest speed at which a free wave of the form (2) can propagate. Therefore, Fig. 3 may also be used to visualize the effect of N upon v_{cr}.

To show the effect of N upon the traveling deflection profile, equ. (26) was evaluated for $v_o/\left(v_{cr}\big|_{N=0}\right) = 0.2$ and different values of N/N_{cr}. These results are shown in Fig. 5. It may be seen that as the axial compression force N approaches the value $\left(N/N_{cr}\right)_{0.2}$ shown in Fig. 4, the amplitude increases rapidly and the wave length decreases, whereas an increasing tensile force has an opposite effect.

CONCLUSION

It was found that v_{cr} increases with an increasing tension force N, whereas v_{cr} decreases with an increasing compression force; v_{cr} approaching zero when $N \rightarrow N_{cr}$. Hence, in the absence of expansion joints in the rails, the critical velocity v_{cr} may be reduced, by a raise in temperature, to within the operational velocities of trains.

In view of this finding it is expected that an analysis, based on a formulation which does include the inertia and damping of the base, will also exhibit a similar effect of N upon v_{cr}.

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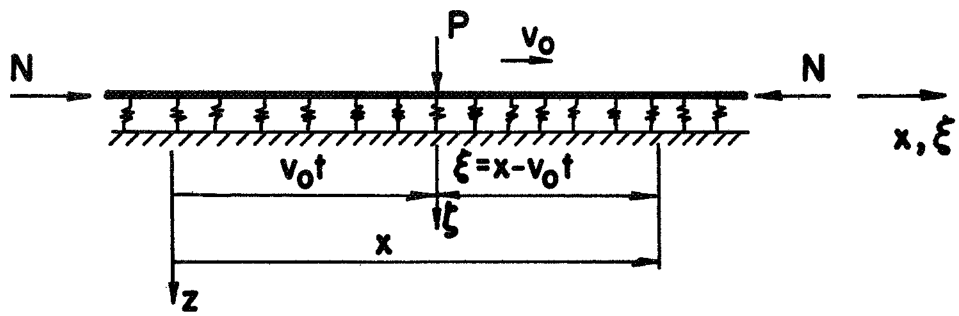

P
v_0
N
N
x, ξ
v_0t
$\xi=x-v_0t$
ζ
x
z

Fig. 1

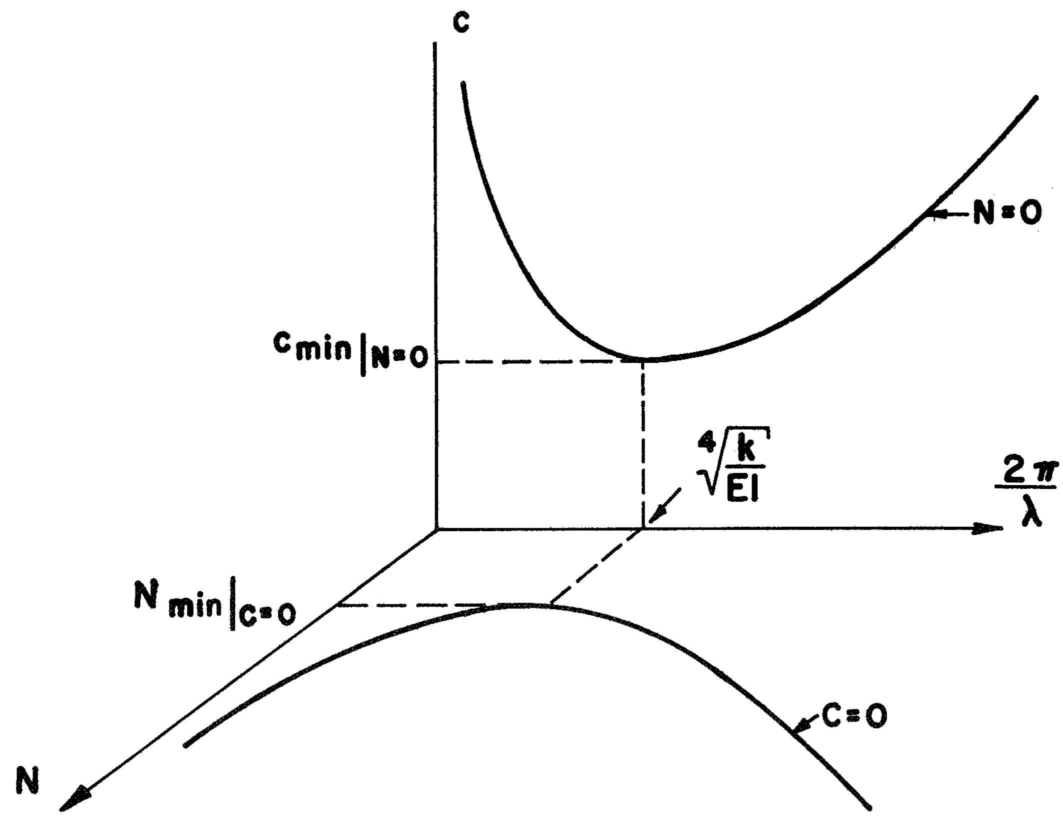
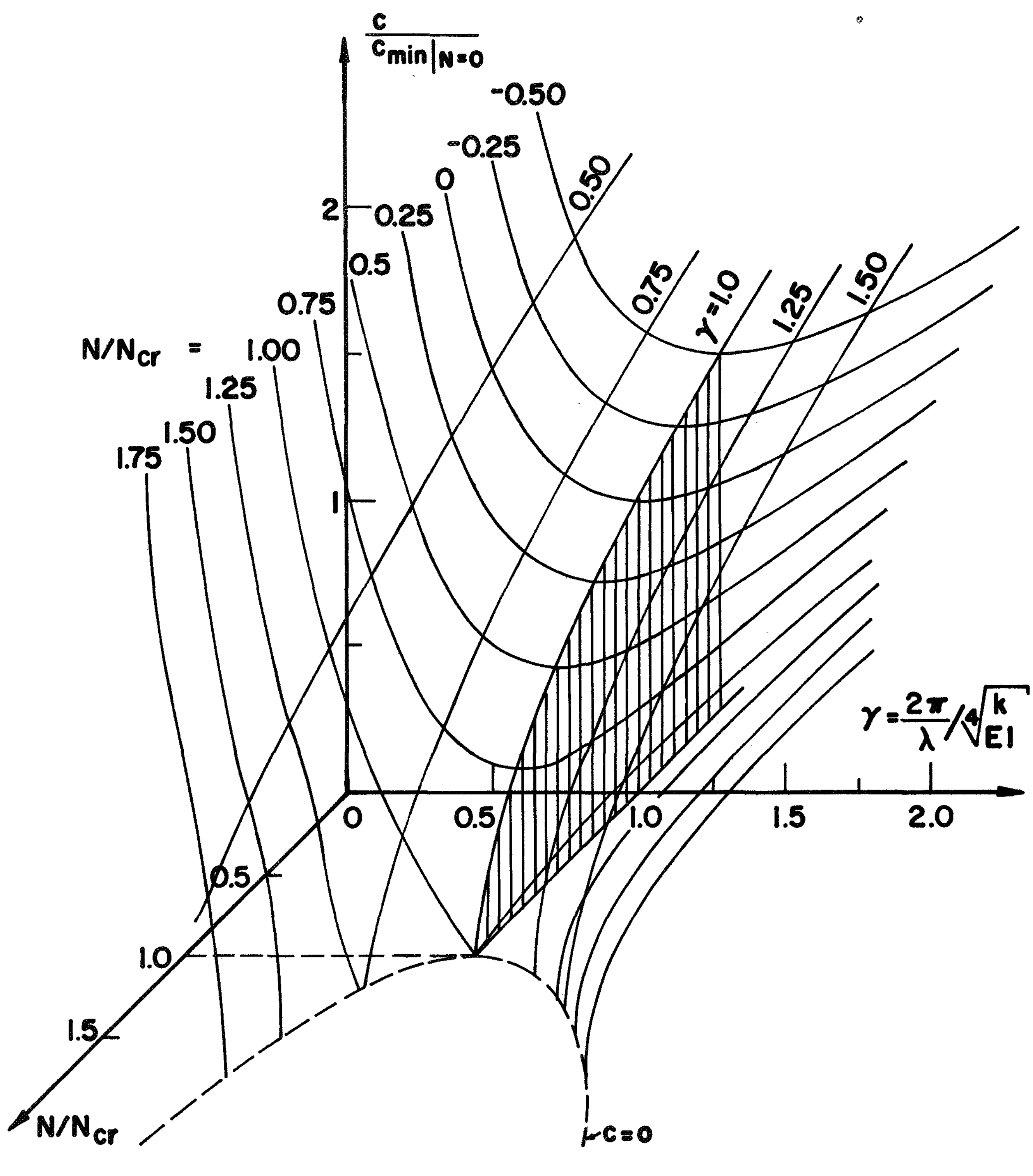
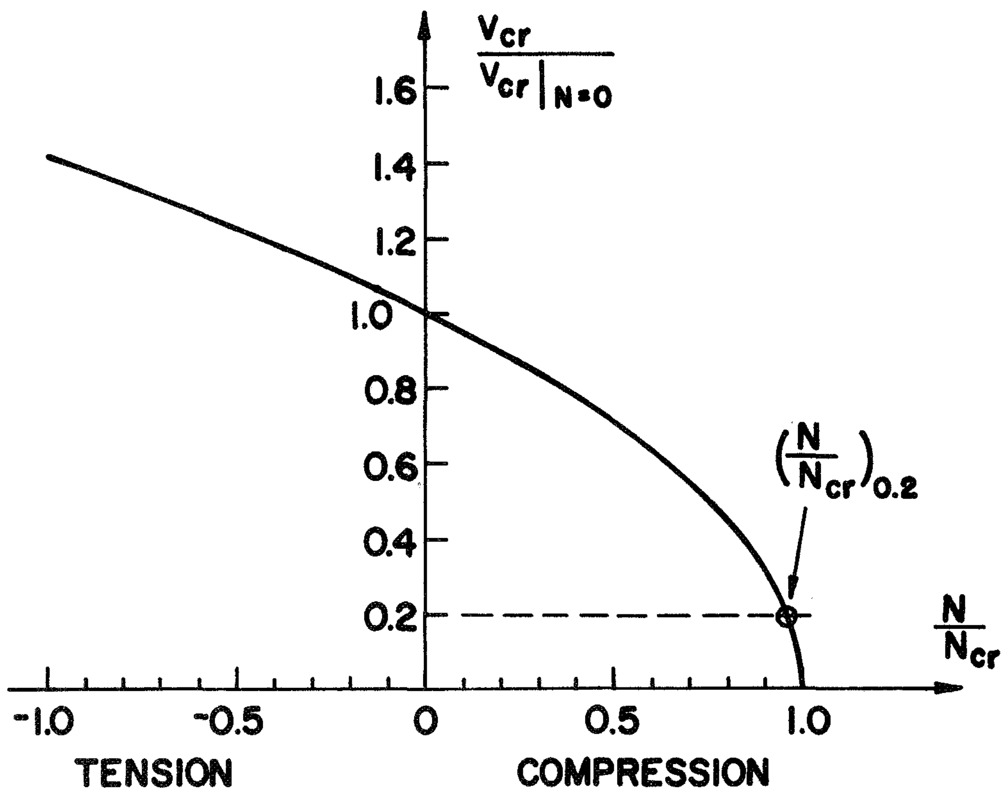

c
N=0
$c_{min}|_{N=0}$
$\sqrt[4]{\frac{k}{EI}}$
$\frac{2\pi}{\lambda}$
$N_{min}|_{C=0}$
C=0
N

Fig. 2



c / c_min|N=0
-0.50
-0.25
0
0.25
0.5
0.75
N/Ncr = 1.00
1.25
1.50
1.75
0.50
0.75
γ=1.0
1.25
1.50
2
1
γ = 2π/λ / ⁴√(k/EI)
0
0.5
1.0
1.5
2.0
0.5
1.0
1.5
N/Ncr
c=0

Fig. 3



$\frac{V_{cr}}{V_{cr}|_{N=0}}$
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
$\left(\frac{N}{N_{cr}}\right)_{0.2}$
$\frac{N}{N_{cr}}$
-1.0
-0.5
0
0.5
1.0
TENSION
COMPRESSION

Fig. 4

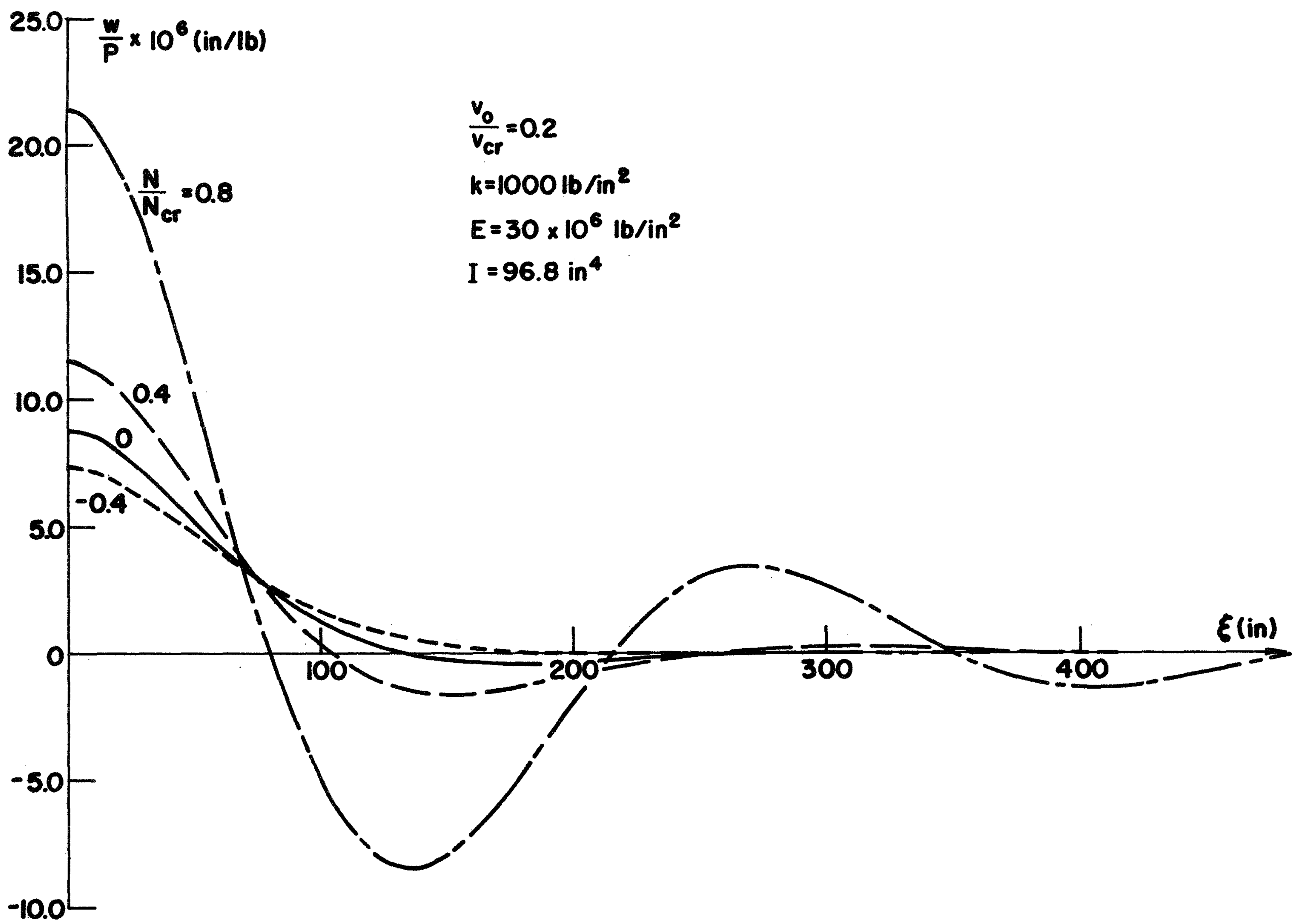

25.0
20.0
15.0
10.0
5.0
0
-5.0
-10.0
$\frac{w}{P} \times 10^6$ (in/lb)
$\frac{N}{N_{cr}} = 0.8$
0.4
0
-0.4
$\frac{V_0}{V_{cr}} = 0.2$
$k = 1000$ lb/in^2
$E = 30 \times 10^6$ lb/in^2
$I = 96.8$ in^4
100
200
300
400
ξ (in)

Fig. 5